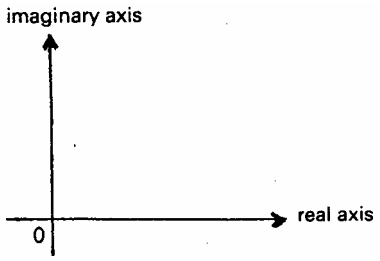
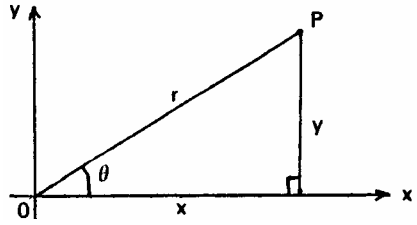
Unit **10: Complex Numbers**

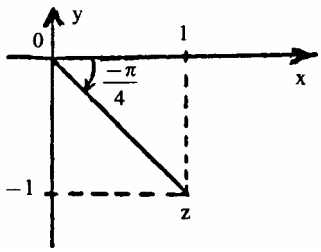
Specific Objectives:

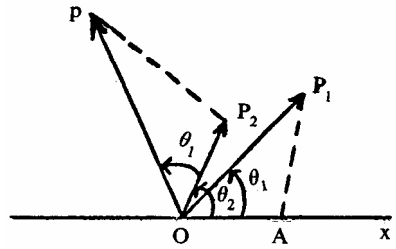
1. To develop the concept of complex numbers.
2. To understand the rectangular and polar forms of complex numbers.
3. To understand the geometric meaning of complex numbers.
4. To understand and apply De Moivre's Theorem.

Detailed Content	Time Ratio	Notes on Teaching
10.1 Complex Number in Rectangular Form	1	By considering the roots of $x^2+1=0$, the imaginary element $i=\sqrt{-1}$ could be introduced. After that teachers may use examples like $x^2+x+1=0$ to introduce the rectangular form $a+bi$. The real part a and the imaginary part b should be clearly stated. Students should know that $a+bi=0$ if and only if $a=b=0$ Noting this definition, teachers could easily show the equality of two complex numbers as $a+bi=c+di$ if and only if $a=c$ and $b=d$
10.2 Addition, Subtraction, Multiplication and Division of Complex Numbers	2	The emphasis here is on the technique and in the case of multiplication students should be alert of the identity $(a+bi)(a-bi)=a^2+b^2$ as $i^2=-1$. Teachers should also mention that the two complex numbers $a\pm bi$ form what is called a conjugate pair and each is the conjugate of the other. It is also useful to note that the property of conjugate pair (i.e. $(a+bi)(a-bi)=a^2+b^2$) could be used in the division. For example, $\frac{a+bi}{c+di}$ could be expressed in rectangular form by multiplying both the numerator and denominator by the conjugate of $c+di$, i.e. $c-di$.

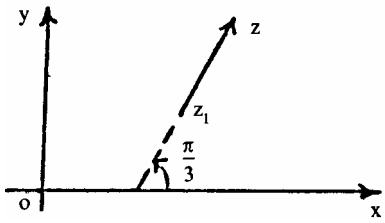
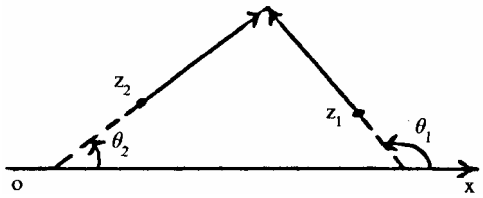
Detailed Content	Time Ratio	Notes on Teaching
		At this stage, teachers may introduce the notation of using a single letter z to represent a complex number $x + yi$ and writing a bar cover the letter z to denote its conjugate, i.e. $\overline{z}=x-yi$. Hence $z+\overline{z}=2x$ and $z-\overline{z}=2yi$. The following properties and facts should be introduced. (1) If z_1 and z_2 are two complex numbers, then (i) $\overline{z_1 \pm z_2}=\overline{z_1}\pm\overline{z_2}$ (ii) $\overline{z_1 \cdot z_2}=\overline{z_1}\cdot\overline{z_2}$ (iii) $\overline{\left(\frac{z_1}{z_2}\right)}=\frac{\overline{z_1}}{\overline{z_2}}$ where $z_2\neq 0$ (2) If $z = p + qi$ is a complex root of the quadratic equation $ax^2+bx+c=0$ where a, b and c are real numbers, then $\overline{z}=p-qi$ is also a root of the equation. (3) If $f(x)$ is a polynomial with real coefficients and $z = p+qi$ is a complex root of the equation $f(x)=0$, then $\overline{z}=p-qi$ is also a root of the equation.
10.3 Complex Numbers in Polar Form	2	Teachers can lead students to represent a complex number in an Argand plane. The real axis and the imaginary axis should be stated. 

Detailed Content	Time Ratio	Notes on Teaching
		Students should be able to transform any complex number z from rectangular form $x + yi$ to polar form $r(\cos\theta + i\sin\theta)$ and vice versa. The relevant facts are: $z = x + yi = r(\cos\theta + i\sin\theta)$ $x = r\cos\theta$, $y = r\sin\theta$ or $r = x^2 + y^2$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ and the value of θ is determined by the signs of x and y. If z is represented by the point P in the Argand Diagram, students should note that r is called the modulus of z. r is denoted by $\|z\|$ and is non-negative. θ is called the argument of z, and is denoted by arg z.  Teachers should clearly stated that θ is, in general, not uniquely defined, i.e. $\theta = \tan^{-1}\left(\frac{y}{x}\right) + 2n\pi$ (n = 0, ±1, ±2, …) To avoid ambiguity, its principal value (i.e. value in the range $-\pi < \theta \le \pi$ is usually adopted in solving relevant problems. The notation r cis θ is a short form for $r(\cos\theta + i\sin\theta)$ and could be introduced for abler students. The following examples or exercises should be useful: *Example 1* If $z = 2 - 2i$, then $\|z\| = 2\sqrt{2}$ and the principal value of arg z is $-\frac{\pi}{4}$.

Detailed Content	Time Ratio	Notes on Teaching
		Teachers may point out that a quick sketch of the position of a point on an Argand diagram will avoid errors in writing down the modulus and argument.  *Example 2* Show that $\|\bar{z}\| = \|z\|$, $\arg\bar{z} = -\arg z$ and $z\bar{z} = \|z\|^2$. It is a useful way of expressing modulus in terms of the complex number and its conjugate. Teachers should emphasize that the modulus squared is the sum of the squares of the coordinates of a complex number on the Argand diagram and is not $x^2 + (iy)^2$. *Example 3* If $z_1 = x_1 + \bar{y}y_1$ and $z_2 = x_2 + \bar{y}y_2$ are represented by P_1 and P_2 respectively on the Argand diagram, then $\frac{z_1 + z_2}{2}$ is represented by the mid point of P_1 and P_2 and if $z_1 + z_2$ is represented by P_3 then $OP_1P_2P_3$ is a parallelogram. Students should notice that the parallelogram law of addition of vectors also applies to complex numbers. They may relate each complex number to the vector joining the origin to the point representing the number on the Argand diagram. Addition and subtraction of complex numbers may be represented geometrically by vector addition and subtraction.

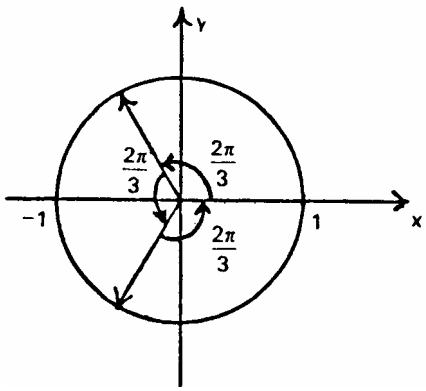
Detailed Content	Time Ratio	Notes on Teaching
10.4 Multiplication and Division of complex numbers in Polar Form	2	Teachers may lead students to discover that the product of two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ can be expressed in the form $r_1r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$, the standard form of a complex number of modulus r_1r_2 and argument $\theta_1 + \theta_2$. Teachers may summarize the results as $\lvert z_1z_2\rvert = \lvert z_1\rvert\lvert z_2\rvert$ and $\arg(z_1z_2) = \arg z_1 + \arg z_2$. Students should be able to apply them to the multiplication of several complex numbers in polar form. They are expected to see that when a given number is multiplied by $r(\cos\theta + i\sin\theta)$, its modulus is multiplied by r and its argument is increased by θ. Geometrically $z = z_1z_2$ can be illustrated on the Argand diagram in which 0 is  the origin, A is the point representing 1 and triangles OAP_1 and OP_2P are similar. In particular, multiplying a number by i will turn the vector representation of the number on the Argand diagram through one right angle without changing its length. The result of dividing z_1 by z_2 may be obtained by considering $z_1 = z_2z$ or $r_1(\cos\theta_1 + i\sin\theta_1) = r_2(\cos\theta_2 + i\sin\theta_2)r(\cos\theta + i\sin\theta)$ $= r_2r\left[\cos(\theta_2+\theta) + i\sin(\theta_2+\theta)\right]$ It is not difficult to see that $\lvert z\rvert = r = \frac{r_1}{r_2}$ and

Detailed Content	Time Ratio	Notes on Teaching
		$\arg z = \theta = \theta_1 - \theta_2$ i.e. $\left\lvert\frac{z_1}{z_2}\right\rvert = \frac{\lvert z_1\rvert}{\lvert z_2\rvert}$ and $\arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$。 *Example* Suppose the numbers 0, 1, z_1, z_2, $z_3$are represented by points O, A, Z_1, Z_2, Z_3 respectively, on an Argand diagram and the triangles OZ_2Z_3, OZ_1A are similar. Show that $z_3 = \frac{z_2}{z_1}$.
10.5 Simple Loci and Complex Numbers	4	Students are also expected to recognize some simple loci. For example: (1) $\lvert z+1\rvert = \lvert z-3\rvert$ is the perpendicular bisector of the line segment joining the points $z = -1$ and $z = 3$. In fact it is helpful for teachers to remind students to put $z = x + iy$ into the equation and obtain a relation of x and y, which is the equation of the locus, i.e. $\lvert x+iy+1\rvert = \lvert x+iy-3\rvert$ $\sqrt{(x+1)^2+y^2} = \sqrt{(x-3)^2+y^2}$ $(x+1)^2+y^2 = (x-3)^2+y^2$ $2x+1 = -6x+9$ or $x = 1$ Students should be able to see that $\lvert z+1\rvert = \lvert z-3\rvert$ can be interpreted as the distance of z from the point -1 being the same as the distance of z from the point representing 3. (2) $\lvert z - z_1\rvert = 4$ is the circle with centre at z_1 and radius 4 units. (3) $\lvert z-2\rvert + \lvert z+2\rvert = 6$ is an ellipse with foci at 2 and -2. (4) $\big\lvert\, \lvert z-5i\rvert - \lvert z+5i\rvert \,\big\rvert = 6$ is a hyperbola with foci at $5i$ and $-5i$. (5) $\arg z = \pi$ is the negative real axis. (6) $\arg(z - z_1) = \frac{\pi}{3}$ is a ray from the point z_1 in the direction $\frac{\pi}{3}$.

Detailed Content	Time Ratio	Notes on Teaching
		Teachers should point out that the vector $z-z_1$ makes an angle of $\frac{\pi}{3}$ with the real axis and that z cannot lie in the other direction of the line through z_1 because the direction of the vector $z-z_1$ would then be $-\frac{2\pi}{3}$.  (7) $\arg\left(\frac{z-z_1}{z-z_2}\right)=\frac{\pi}{2}$ may be written as $\arg(z-z_1)-\arg(z-z_2)=\frac{\pi}{2}$  putting $\arg(z-z_1)=\theta_1$ and $\arg(z-z_2)=\theta_2$, $\theta_1-\theta_2=$ angle subtended by the line segment joining z_1 and z_2. $\therefore$ z lies on the semicircle with the line segment (with end points z_1 and z_2) as diameter.
10.6 De Moivre's Theorem	3	Teachers should guide students to prove $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$ for all positive integral value of n. The Principle of Mathematical Induction could be used.

Detailed Content	Time Ratio	Notes on Teaching
		Example 1 For $\cos 3\theta+i\sin 3\theta=(\cos\theta+i\sin\theta)^3$, by expanding the right hand side by the binomial theorem and equating real parts and imaginary parts, students should be able to obtain $\cos 3\theta=4\cos^3\theta-3\cos\theta$ and $\sin 3\theta=3\sin\theta-4\sin^3\theta$. *Example* 2 If $z=\cos\theta+i\sin\theta$ show that $z^n-\frac{1}{z^n}=2i\sin n\theta$ Hence express $\sin^5\theta$ in terms of trigonometrical ratios of multiple angles. By considering $(2i\sin\theta)^5=\left(z-\frac{1}{z}\right)^5$ students should be able to obtain $2^5 i\sin^5\theta$ $=\left(z^5-\frac{1}{z^5}\right)-5\left(z^3-\frac{1}{z^3}\right)+10\left(z-\frac{1}{z}\right)$ or $\sin^5\theta=\frac{1}{16}(\sin 5\theta-5\sin 3\theta+10\sin\theta)$ Students should realize that the two identities $z^n+\frac{1}{z^n}=2\cos n\theta$ and $z^n-\frac{1}{z^n}=2i\sin n\theta$ are very useful in solving similar problems. De Moivre's Theorem could be extended to the case when n is negative and when n is rational. . When n is a negative integer, then $m=-n$ is a positive integer and $(\cos\theta+i\sin\theta)^n=(\cos\theta+i\sin\theta)^{-m}$ $=\frac{1}{(\cos\theta+i\sin\theta)^m}=\frac{1}{(\cos m\theta+i\sin m\theta)}$

Detailed Content	Time Ratio	Notes on Teaching
		$= \dfrac{\cos m\theta - i\sin m\theta}{1} = \cos(-m\theta) + i\sin(-m\theta)$ $= \cos n\theta + i\sin n\theta$ showing that the De Moivre's Theorem is also true. When n is a fraction, it can be written as $\dfrac{p}{q}$ where p and q are integers and q is positive. It would involve taking a qth root. Teachers should emphasize that in complex numbers there will be q different values. $\cos\dfrac{p\theta}{q} + i\sin\dfrac{p\theta}{q}$ is one value of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ and more generally, the values of $(\cos\theta + i\sin\theta)^{\frac{p}{q}}$ are $\cos\left(\dfrac{p\theta}{q} + \dfrac{2k\pi}{q}\right) + i\sin\left(\dfrac{p\theta}{q} + \dfrac{2k\pi}{q}\right)$ for k = 0, 1, 2, … , q – 1.
10.7 The nth roots of Unity	3	The cube roots of unity is an interesting example. $z^3 = 1 = \cos 2k\pi + i\sin 2k\pi$ for k = 0, 1, 2 $z = 1$ or $\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$ or $\left(\cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}\right)$ = 1 or $-\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$ or $-\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$ The cube roots of unity may be denoted by 1, ω and ω^2 and its properties may be discussed: (1) $\omega^3 = 1$ (2) $1 + \omega + \omega^2 = 0$ (3) $(\omega^2)^2 = \omega^4 = \omega$, i.e. each complex root is the square of the other. (4) $1 = \omega^3 = \omega^6 = \cdots$, $\omega = \omega^4 = \omega^7 = \cdots$, and $\omega^2 = \omega^5 = \omega^8 = \cdots$,

Detailed Content	Time Ratio	Notes on Teaching
		The cube roots of unity may also be mapped on the Argand diagram as follows:  Similarly the nth roots of unity are $z = \cos\dfrac{2k\pi}{n} + i\sin\dfrac{2k\pi}{n}$ for k = 0, 1, 2, … , n – 1 The following should be discussed: (1) Giving k further values does not give further roots, as k = n gives the argument 2π, which provides the same results as k = 0, k = n + 1 repeats k = 1 and so on. (2) All the roots have modulus equal to 1 and their representations in an Argand diagram will be equally spaced round the unit circle, the difference between successive arguments being $\dfrac{2\pi}{n}$.

Detailed Content	Time Ratio	Notes on Teaching
		(3) These mappings are symmetrical about the real axis, so the complex roots are in conjugate pairs. (4) If n is even, one value of k is $\frac{n}{2}$ giving the argument π and the corresponding real root $z = -1$. In this case $z^2 - 1 = (z-1)(z+1)$ is a quadratic factor of $z^n - 1$. (5) If n is odd, $z = 1$ is the only real root and $(z - 1)$ the only real factor. (6) Any positive real number a can be written as $a(\cos 2k\pi + i\sin 2k\pi)$ and its nth roots will be $\sqrt[n]{a}\left(\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}\right)$ where $\sqrt[n]{a}$ is the positive nth root of a. *Example* The fifth roots of 32 are 2, $2\left(\cos\frac{2\pi}{5} \pm i\sin\frac{2\pi}{5}\right)$, $2\left(\cos\frac{4\pi}{5} \pm i\sin\frac{4\pi}{5}\right)$ Students should be able to apply De Moivre's Theorem to find the nth root of a complex number. For example, if $z = r(\cos\theta + i\sin\theta)$ then $z^{\frac{1}{n}} = r^{\frac{1}{n}}\left(\cos\frac{\theta + 2k\pi}{n} + i\sin\frac{\theta + 2k\pi}{n}\right)$ where $k = 0, 1, 2, \ldots, n - 1$.
	17	